1) WRITE OUT THE STEPS BETWEEN MT5'S EQUATIONS 4.28 AND 4.29 AS SHOWN.

$$\tau = 4\sqrt{\frac{l}{g}}\int_0^1 [(1-z^2)(1-k^2z^2)]^{-1/2}dz \qquad (4.28)$$

Numerical values for integrals of this type can be found in various tables.

For oscillatory motion to result, $|\theta_0| < \pi$, or, equivalently, $\sin(\theta_0/2) = k$, where $-1 < k < +1$. For this case, we can evaluate the integral in Equation 4.28 by expanding $(1 - k^2z^2)^{-1/2}$ in a power series:

$$(1-k^2z^2)^{-1/2} = 1 + \frac{k^2z^2}{2} + \frac{3k^4z^4}{8} + \cdots \qquad \textbf{(4.28b)}$$

Then, the expression for the period becomes

$$\tau = 4\sqrt{\frac{l}{g}}\int_0^1 \frac{dz}{(1-z^2)^{1/2}}\left[1 + \frac{k^2z^2}{2} + \frac{3k^4z^4}{8} + \cdots\right] \qquad \textbf{(4.28c)}$$

$$= 4\sqrt{\frac{l}{g}}\left[\frac{\pi}{2} + \frac{k^2}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \frac{3k^4}{8}\cdot\frac{3}{8}\cdot\frac{\pi}{2} + \cdots\right] \qquad \textbf{(4.28d)}$$

$$= 2\pi\sqrt{\frac{l}{g}}\left[1 + \frac{k^2}{4} + \frac{9k^4}{64} + \cdots\right] \qquad \textbf{(4.28e)}$$

If $|k|$ is large (i.e., near 1), then we need many terms to produce a reasonably accurate result. But for small k, the expansion converges rapidly. And because $k = \sin(\theta_0/2)$, then $k \cong (\theta_0/2) - (\theta_0^3/48)$; the result, correct to the fourth order, is

$$\tau \cong 2\pi\sqrt{\frac{l}{g}}\left[1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4\right] \qquad (4.29)$$

4.28 IS

$$\tau = 4\sqrt{\frac{l}{g}}\int_0^1 [(1-z^2)(1-k^2z^2)]^{-1/2} dz$$

THE FIRST STEP IS TO EXPAND THE BINOMIAL

$$(1-k^2z^2)^{-\frac{1}{2}}$$

FROM APPENDIX D (EQUATION D.6)

$$(1 \pm x)^{-1/2} = 1 \mp \frac{x}{2} + \frac{3x^2}{8} \mp \frac{5x^3}{16} + \ldots$$

THUS

$$(1-k^2z^2)^{-1/2} = 1 + \frac{k^2z^2}{2} + \frac{3k^4z^4}{8} + \ldots \qquad (4.28b)$$

SUBSTITUTING THIS INTO 4.28 GIVES 4.28c

$$\tau = 4\sqrt{\frac{l}{g}}\int_0^1 \frac{dz}{\sqrt{1-z^2}}\left[1 + \frac{k^2z^2}{2} + \frac{3k^4z^4}{8} + \ldots\right] \qquad (4.28c)$$

EXPANDING THIS GIVES

$$\tau = 4\sqrt{\frac{l}{g}}\left\{\int_0^1 \frac{dz}{\sqrt{1-z^2}} + \frac{k^2}{2}\int_0^1 \frac{z^2dz}{\sqrt{1-z^2}} + \frac{3k^4}{8}\int_0^1 \frac{z^4dz}{\sqrt{1-z^2}} + \ldots\right\}$$

APPENDIX E.3 GIVES SOLUTIONS OF THESE INTEGRALS IN TERMS OF Γ FUNCTIONS

$$\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n}\frac{\Gamma(\frac{1}{n})}{\Gamma(\frac{1}{n}+\frac{1}{2})} \longrightarrow \int_0^1 \frac{dz}{\sqrt{1-z^2}} \qquad (E.26)$$

FOR THE FIRST INTEGRAL, $n=2$

$$\int_0^1 \frac{dz}{\sqrt{1-z^2}} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2})} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{1}{2})}{\Gamma(1)} \longrightarrow$$

1) CONTINUED

THE VALUES OF THE Γ FUNCTIONS ARE GIVEN ON p 615

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(1) = 1$$

THUS

$$\int_0^1 \frac{dz}{\sqrt{1-z^2}} = \frac{\sqrt{\pi}}{2}\frac{\sqrt{\pi}}{1} = \boxed{\frac{\pi}{2} = \int_0^1 \frac{dz}{\sqrt{1-z^2}}} \quad 1^{ST} \text{ INTEGRAL}$$

FOR THE 2^{ND} & 3^{RD} INTEGRALS US (E.27a)

$$\int_0^1 x^m (1-x^2)^n dx = \frac{\Gamma(n+1)\,\Gamma\left(\frac{m+1}{2}\right)}{2\Gamma\left(n+\frac{m+3}{2}\right)} \qquad (E.27a)$$

FOR

$$\int_0^1 \frac{z^2 dz}{\sqrt{1-z^2}} \Rightarrow m=2,\ n=-\tfrac{1}{2},\ \frac{m+1}{2}=\frac{3}{2},\ n+\frac{m+3}{2}=\frac{4}{2}=2$$

$$\int_0^1 \frac{z^2 dz}{\sqrt{1-z^2}} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma(2)}$$

NOTING EQUATION E.20: $n\Gamma(n) = \Gamma(n+1)$ GIVES

$$\Gamma\left(\tfrac{3}{2}\right) = \Gamma\left(\tfrac{1}{2}+1\right) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\left(\sqrt{\pi}\right) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma(2) = 1 \qquad (E.25)$$

GIVES

$$\int_0^1 \frac{z^2 dz}{\sqrt{1-z^2}} = \frac{\left(\sqrt{\pi}\right)\left(\frac{\sqrt{\pi}}{2}\right)}{2(1)} = \boxed{\frac{\pi}{4} = \int_0^1 \frac{z^2 dz}{\sqrt{1-z^2}}} \quad 2^{ND} \text{ INTEGRAL}$$

FOR THE LAST INTEGRAL

$$\int_0^1 \frac{z^4 dz}{\sqrt{1-z^2}} \Rightarrow m=4,\ n=-\tfrac{1}{2},\ \frac{m+1}{2}=\frac{5}{2},\ \underbrace{n+\frac{m+3}{2}}_{-\frac{1}{2}+\frac{7}{2}}=\frac{6}{2}=3$$

$$\int_0^1 \frac{z^4 dz}{\sqrt{1-z^2}} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{2\Gamma(3)}$$

$$\Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{2}\Gamma\left(\tfrac{3}{2}\right) = \left(\tfrac{3}{2}\right)\left(\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}\right) = \tfrac{3}{4}\sqrt{\pi}$$

$$\Gamma(3) = 2\Gamma(2) = 2(1) = 2$$

1) CONTINUED

$$\int_0^1 \frac{z^4\,dz}{\sqrt{1-z^2}} = \frac{(\sqrt{\pi})(\frac{3}{4})(\sqrt{\pi})}{2(2)} = \boxed{\frac{3\pi}{16} = \int_0^1 \frac{z^4\,dz}{\sqrt{1-z^2}}}$$

BACK TO THE EQUATION

$$T = \cancel{4}^{2}\sqrt{\frac{l}{g}}\left\{\frac{\pi}{\cancel{2}} + \frac{k^2}{\cancel{2}}\left(\frac{\pi}{4}\right) + \frac{3k^4}{\cancel{8}_4}\left(\frac{3\pi}{16}\right) + \dots\right\} \quad (4.28d)$$

REARRANGING

$$T = 2\pi\sqrt{\frac{l}{g}}\left\{1 + \frac{k^2}{4} + \frac{9k^4}{64} + \dots\right\} \quad (4.28e)$$

NOW FOR THE LAST STEP, EXPAND k USING APPENDIX D

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad (D.28)$$

THUS

$$k = \sin\left(\frac{\theta_0}{2}\right) = \left(\frac{\theta_0}{2}\right) - \frac{1}{6}\left(\frac{\theta_0}{2}\right)^3 + \dots \qquad 2^3 = 8$$

$$k \approx \frac{\theta_0}{2} - \frac{\theta_0^3}{48} + \dots$$

$$k^2 \approx \frac{\theta_0^2}{4} - 2\frac{\theta_0\theta_0^3}{(2)(48)} + \frac{\theta_0^6}{(48)^2} \approx \frac{\theta_0^2}{4} - \frac{\theta_0^4}{48}$$

$$k^4 \approx \frac{\theta_0^4}{16} - \text{TERMS WITH } \theta_0 \text{ POWERS} > 4$$

SUBSTITUTING INTO (4.28e) NEGLECTING θ_0 POWERS > 4

$$T = 2\pi\sqrt{\frac{l}{g}}\left\{1 + \frac{1}{4}\left(\frac{\theta_0^2}{4} - \frac{\theta_0^4}{48}\right) + \frac{9}{64}\left(\frac{\theta_0^4}{16}\right)\right\} \qquad 4(48) = 2^6 \cdot 3 \qquad (64)(16) = 2^{10}$$

$$T = 2\pi\sqrt{\frac{l}{g}}\left\{1 + \frac{\theta_0^2}{16} - \frac{\theta_0^4}{4(48)} + \frac{9\theta_0^4}{(64)(16)}\right\} \qquad \Rightarrow \text{LCD} = 2^{10}\cdot 3 = 3072$$

$$\left(-\frac{1}{192} + \frac{9}{1024}\right)\theta_0^4 = \left(-\frac{16}{3072} + \frac{27}{3072}\right)$$

THUS

$$\boxed{T = 2\pi\sqrt{\frac{l}{g}}\left\{1 + \frac{\theta_0^2}{16} + \frac{11\theta_0^4}{3072}\right\}}$$

QED!